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Gravitating semirelativistic N-boson systems

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Abstract

Analytic energy bounds for N-boson systems governed by semirelativistic Hamiltonians of the form $H = \sum_{i=1}^{N} \left(\mathbf{p}_i^2 + m^2\right)^{1/2} - \sum_{1=i<j}^{N} v/r_{ij}$, with $v > 0$, are derived by use of Jacobi relative coordinates. For gravity $v = c/N$, these bounds are substantially tighter than earlier bounds and they are shown to coincide with known results in the nonrelativistic limit.

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1. Introduction: the N-body problem

One-body Hamiltonians H composed of the relativistic expression $\sqrt{\mathbf{p}^2 + m^2}$ for the kinetic energy of particles of mass m and momentum $\mathbf{p}$ and of a coordinate-dependent static interaction potential $V(\mathbf{r})$, defined as operator sum

$$H = \sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{r}),$$

provide a simple but very efficient tool for the description of relativistically moving particles [1–6]. They have been used, for instance, for the description of hadrons as bound states of quarks [7, 8]. One of the advantages of this kind of semirelativistic treatment is that its generalization to the many-body problem is straightforward to formulate. A semirelativistic Hamiltonian for a system of N identical particles interacting by pair potentials $V(r_{ij})$ is given by

$$H = \sum_{i=1}^{N} \sqrt{\mathbf{p}_i^2 + m^2} + \sum_{1=i<j}^{N} V(r_{ij}). \tag{1}$$

We shall use the notational simplification $p \equiv |\mathbf{p}|$, $r \equiv |\mathbf{r}|$ or $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$, whenever no ambiguity is introduced by doing so. Many approaches to such many-body problems for identical particles employ the very powerful constraint of permutation symmetry to generate their reduction to a two-body problem with a Hamiltonian $\mathcal{H}$ whose spectrum is used to approximate the many-body energy eigenvalues or to generate a lower energy bound. This reduction may be effected in various ways, which leads to the problem of finding the most effective reduced problem, the one which would provide the *highest* lower bound. In this paper, we use Jacobi relative coordinates which introduce technical difficulties but yield a lower bound that is much improved over previous results. We have already achieved a similar improvement for the harmonic-oscillator potential [9] and for its convex transformations [10]. Here, we show that one can derive tight N-body energy bounds for the (more physically important) case of the gravitational potential: $V(r) = -v/r$, $v > 0$. There is a long history of attention to the corresponding nonrelativistic problem, dating back at least to 1967 [11, 12]. Also the semirelativistic case has been discussed before, notably by Lieb and Thirring [13], by Lieb and Yau [14], as well as by Martin and Roy [15].

In their rigorous investigation [14] of Chandrasekhar's theory of the stellar collapse and its formal connection to quantum physics, Lieb and Yau were able to demonstrate exactly that in the simultaneous limit $N \to \infty$ and $v \to 0$ of particle number N and gravitational coupling constant v such that their product Nv is kept fixed (at an arbitrary value below some critical value) the Schrödinger equation reduces to the 'semiclassical' approximation represented by a Hartree-type equation for the density of the bosons. In particular, they succeeded in proving the convergence of the lowest quantum energy, E, defined as the infimum of the spectrum of the Hamiltonian H given by equation (1), to the corresponding semiclassical minimum Hartree energy. Interestingly, precisely such kind of behaviour in the limit $N \to \infty$ has been conjectured already before in [13].

The study in [15] shows that for the one-body case with a gravitational potential a lower bound to the energy spectrum may be expressed in terms of the lowest energy of a Schrödinger operator with a Kratzer potential, which is a Coulomb potential 'spiked' with an additional term of the form A/r^2. In spite of the complications of relative coordinates but with boson permutation symmetry in the individual-particle coordinates, we may take advantage of this result to construct improved lower energy bounds for systems comprising identical particles with integer spin.

The plan of this paper is as follows. After recalling, in section 2, existing Coulomb one-body lower energy bounds, and analysing, in section 3, the simple $(N/2)$ N-body lower energy bound, we reformulate, in section 4, the N-particle problem in relative coordinates. This allows us to derive, in section 5, an improved lower energy bound. To this result we adjoin, in section 6, a variational (Rayleigh–Ritz) upper energy bound obtained with the help of a scale-optimized Gaussian trial wavefunction. We summarize our results in section 7, and inspect, in particular, their large-N limit.

2. One-body lower energy bounds for the relativistic Coulomb problem

The case of a one-particle Hamiltonian with a Coulomb potential $V(r) = -v/r$ has been extensively investigated. We know from the pioneering work of Herbst [16] that the Hamiltonian is self-adjoint for $v \leqslant \frac{1}{2}$ and has a Friedrichs extension up to the critical value $v = 2/\pi$. Herbst obtains the following lower bound to the ground-state energy E:

$$E \geqslant m\sqrt{1 - \left(\frac{\pi v}{2}\right)^2}, \qquad 0 \leqslant v < \frac{2}{\pi}.$$

For the smaller range $0 \leqslant v < \frac{1}{2}$ of the Coulomb coupling v, this result was strengthened by

Martin and Roy [15] to

$$E \geqslant m\sqrt{\frac{1+\sqrt{1-(2v)^2}}{2}}, \qquad 0 \leqslant v < \frac{1}{2}. \tag{2}$$

This energy bound was found by considering the squares of the kinetic-energy and the potential-energy terms and by relating certain expectations to the previously studied exact solution of the Klein–Gordon Schrödinger equation. Let us now briefly re-derive this same result, emphasizing that this lower bound is, in fact, generated by the lowest energy of a Schrödinger operator [17]; this latter problem had earlier been analysed by Kratzer and others [18–24].

We suppose that the exact normalized ground state of H is ψ so that $(p^2+m^2)^{1/2}\psi = (E+v/r)\psi$. By considering the equality between the squares of these equal vectors we obtain, upon introducing a quantity ℓ by $\ell(\ell+1) = -v^2$,

$$\begin{aligned} E^2 - m^2 &= \left(\psi, \left(p^2 - \frac{2Ev}{r} - \frac{v^2}{r^2}\right)\psi\right) \\ &\geqslant -\frac{E^2v^2}{(\ell+1)^2}; \end{aligned} \tag{3}$$

the inequality on the right-hand side of (3) arises from the variational principle applied to the Kratzer Hamiltonian:

$$H_K = p^2 - \frac{2Ev}{r} + \frac{\ell(\ell+1)}{r^2} = p^2 - \frac{2Ev}{r} - \frac{v^2}{r^2}.$$

The well-known expression for the bottom, for given ℓ, of the hydrogen energy spectrum remains valid for negative non-integer ℓ provided $|\ell| < \frac{1}{2}$, which is equivalent to the constraint $v < \frac{1}{2}$. By solving for ℓ in terms of the coupling parameter v, we recover the Martin–Roy lower bound (2).

3. 'Simple' (or $N/2$) lower energy bound for general N-body problems

This rather simple lower energy bound is not limited to the gravitational pair potential, and it allows us to prove that a given N-body Hamiltonian is bounded from below. Applying the same reasoning to the (soluble) Schrödinger harmonic-oscillator problem, defined by the Hamiltonian

$$H = \sum_{i=1}^{N} \mathbf{p}_i^2 + \sum_{1=i<j}^{N} r_{ij}^2,$$

with the exact ground-state energy $E = 3(N-1)\sqrt{N}$, one gets only $E/\sqrt{2}$ whereas a general lower bound based on Jacobi relative coordinates yields indeed the exact energy for this potential [25]. A lower bound to the ground-state energy of the Hamiltonian (1) is provided by the bottom $\mathcal{E}_{N/2}$ of the spectrum of the one-body Hamiltonian operator

$$\mathcal{H}_{N/2} = N\left[\sqrt{p^2+m^2} + \frac{N-1}{2}V(r)\right], \tag{4}$$

since boson permutation symmetry of the exact N-body normalized wavefunction Ψ implies $(\Psi, H\Psi) = (\Psi, h\Psi)$, where the operator h is a two-body Hamiltonian given by

$$h = \frac{N}{2}\left[\sqrt{\mathbf{p}_1^2+m^2} + \sqrt{\mathbf{p}_2^2+m^2} + (N-1)V(r_{12})\right].$$

Changing the coordinates of this two-particle problem to $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2$, the individual momenta are given, in terms of the corresponding total and relative momenta variables, by $\mathbf{p}_{1,2} = \mathbf{P} \pm \mathbf{p}$. Using the lemma of [9] to remove the centre-of-mass momentum term $\mathbf{P}$, the reduced two-body Hamiltonian h becomes $\mathcal{H}_{N/2}$. If we now apply this simple bound to the gravitational problem, $V(r) = -v/r$, then we find, from (4) and the one-body lower bound (2),

$$E \geqslant Nm\sqrt{\frac{1+\sqrt{1-(N-1)^2v^2}}{2}}, \qquad (N-1)v < 1. \tag{5}$$

4. *N*-particle problems in terms of Jacobi relative coordinates

Our Hamiltonian H does not have the kinetic energy of the N-particle system's centre-of-mass removed. Thus, its eigenvectors are subject to two fundamental symmetries: translation invariance, and boson permutation symmetry (in the individual-particle coordinates $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$. Jacobi relative coordinates may be defined with the aid of an orthogonal matrix B transforming old ($\{\mathbf{r}_i\}$) to new ($\{\boldsymbol{\rho}_i\}$) coordinates by $[\boldsymbol{\rho}] = B[\mathbf{r}]$. The first row of B, with all entries $B_{1i} = 1/\sqrt{N}$, defines a centre-of-mass variable $\boldsymbol{\rho}_1$, its second a pair distance $\boldsymbol{\rho}_2 = (\mathbf{r}_1 - \mathbf{r}_2)/\sqrt{2}$, and the kth row ($k \geqslant 2$) first has $k-1$ entries $B_{ki} = 1/\sqrt{k(k-1)}$, the kth entry $B_{kk} = -\sqrt{(k-1)/k}$, and zero for all remaining entries. The momenta $\{\boldsymbol{\pi}_i\}$ conjugate to the $\{\boldsymbol{\rho}_i\}$ read $[\boldsymbol{\pi}] = (B^{-1})^{\mathrm{T}}[\mathbf{p}] = B[\mathbf{p}]$. Now, for an N-boson problem with an attractive potential $V(r)$, let $\Psi(\boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \ldots, \boldsymbol{\rho}_N)$ be the (still to be found) normalized ground-state eigenfunction corresponding to the lowest energy E. Boson symmetry is a powerful constraint that greatly reduces the complexity of this problem. Although a non-Gaussian wavefunction is not necessarily symmetric in the Jacobi coordinates, we do have the remarkable N-representability expressions [9] (appendix A here) for $i, j > 1$

$$(\Psi, (\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_j)\Psi) = \delta_{ij}\left(\Psi, \boldsymbol{\rho}_2^2\Psi\right) \tag{6}$$

$$(\Psi, (\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j)\Psi) = \delta_{ij}\left(\Psi, \boldsymbol{\pi}_2^2\Psi\right). \tag{7}$$

We now choose to use $\mathbf{p}_N$ and $\mathbf{p}_{N-1}$ and we obtain the reduction

$$E = (\Psi, H\Psi) = \left(\Psi, \left[\frac{N}{2}\sqrt{\mathbf{p}_N^2 + m^2} + \frac{N}{2}\sqrt{\mathbf{p}_{N-1}^2 + m^2} + \gamma V(|\mathbf{r}_{N-1} - \mathbf{r}_N|)\right]\Psi\right),$$

where $\gamma = \frac{1}{2}N(N-1)$. We first note that the last row of B (which defines $\mathbf{p}_N$) is given by

$$[a, a, a, \ldots, a, -(N-1)a], \qquad \text{where} \qquad a = 1/\sqrt{N(N-1)}.$$

Now we express the equation for E in terms of new coordinates defined by the following relations

$$\mathbf{r} = \mathbf{r}_{N-1} - \mathbf{r}_N = \alpha\boldsymbol{\rho}_N - \beta\boldsymbol{\rho}_{N-1}, \qquad r = |\mathbf{r}|,$$

$$\begin{bmatrix}\mathbf{R}\\ \mathbf{r}\end{bmatrix} = \begin{bmatrix}\beta & \alpha\\ \alpha & -\beta\end{bmatrix}\begin{bmatrix}\boldsymbol{\rho}_N\\ \boldsymbol{\rho}_{N-1}\end{bmatrix}, \qquad \begin{bmatrix}\mathbf{P}\\ \mathbf{p}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\beta & \alpha\\ \alpha & -\beta\end{bmatrix}\begin{bmatrix}\boldsymbol{\pi}_N\\ \boldsymbol{\pi}_{N-1}\end{bmatrix},$$

where

$$\alpha = \sqrt{\frac{N}{N-1}} > 1, \qquad \beta = \sqrt{\frac{N-2}{N-1}} < 1, \qquad \delta = \sqrt{\frac{N-2}{N}} < 1.$$

Consequently

$$\alpha^2 + \beta^2 = 2, \qquad a^2 + \beta^2 = \frac{1}{\alpha^2} = \lambda = \frac{N-1}{N},$$

and

$$(N-1)a = \alpha^{-1}, \qquad Na = \alpha, \qquad \delta = \frac{\beta}{\alpha}, \qquad 1+\delta^2 = 2\lambda.$$

We note that because of the boson symmetry, and after removal of π_1, we have generally that

$$\langle f(|\mathbf{p}_{N-1}|)\rangle = \langle f(|\mathbf{p}-\delta\mathbf{P}|)\rangle = \langle f(|\mathbf{p}+\delta\mathbf{P}|)\rangle = \langle f(|\mathbf{p}_N|)\rangle,$$

where $f(p)$ is any appropriate kinetic-energy function. The expression for the energy now has the form

$$E = \left\langle \frac{N}{2}\sqrt{(\mathbf{p}+\delta\mathbf{P})^2+m^2} + \frac{N}{2}\sqrt{(\mathbf{p}-\delta\mathbf{P})^2+m^2} + \gamma V(r)\right\rangle,$$

or, equivalently,

$$E = \langle \mathcal{H}\rangle, \qquad \text{where} \qquad \mathcal{H} = N\sqrt{(\mathbf{p}+\delta\mathbf{P})^2+m^2} + \gamma V(r). \tag{8}$$

The Hamiltonian $\mathcal{H}$ is bounded below by the simple bound $\mathcal{H}_{N/2}$ of equation (4). This result is proved in appendix B. We now consider the eigenequation $\mathcal{H}\psi = \mathcal{E}\psi$, where $\psi(\mathbf{r},\mathbf{R})$ retains some of the boson symmetry implications of the full wavefunction. What we need are immediate consequences of equation (7), namely $\langle \mathbf{p}^2\rangle = \langle \mathbf{P}^2\rangle$, $\langle \mathbf{p}\cdot\mathbf{P}\rangle = 0$, and hence

$$\langle(\mathbf{p}+\delta\mathbf{P})^2\rangle = \langle p^2+\delta^2P^2\rangle = (1+\delta^2)\langle p^2\rangle = 2\lambda\langle p^2\rangle. \tag{9}$$

5. Improved lower energy bound

The lower bound that improves the $N/2$ bound (5) and forms our main result is specifically for the attractive pair potential $V(r) = -v/r, v > 0$. Suppose that $\psi(\mathbf{r},\mathbf{R})$ is the exact lowest eigenfunction of the Hamiltonian $\mathcal{H}$ in (8). We look in a restricted domain $\mathcal{D}$ in the Hilbert space $L^2(R^6)$, one that keeps some of the original boson symmetry, namely, we assume equation (9). This allows us to apply the same reasoning as we did for the one-body problem because the kinetic term is squared and thus the final form of the expectation values involves only the conjugate variables $\{\mathbf{r},\mathbf{p}\}$. Let $\mathcal{E}$ be the minimum of $(\psi,\mathcal{H}\psi)$ corresponding to normalized ψ satisfying (9). $\mathcal{E}$ is a lower bound to $E, \mathcal{E} \leqslant E$, because, if Ψ is the exact normalized N-body wavefunction, then

$$E = (\Psi, H\Psi) = (\Psi,\mathcal{H}\Psi) \geqslant (\psi,\mathcal{H}\psi) = \mathcal{E}.$$

The eigenvalue equation of $\mathcal{H}, \mathcal{H}\psi = \mathcal{E}\psi$, explicitly reads

$$N\sqrt{(\mathbf{p}+\delta\mathbf{P})^2+m^2}\psi = \left(\mathcal{E}+\frac{\gamma v}{r}\right)\psi.$$

Squaring these equal vectors and use of (9), and also the scaling change $\{r,p\}\to\{2r,p/2\}$, implies

$$\mathcal{E}^2 - N^2m^2 \geqslant \gamma \inf_{\psi\in\mathcal{D}}\left(\psi, \left(p^2 - \frac{v\mathcal{E}}{r} - \frac{\gamma v^2}{4r^2}\right)\psi\right).$$

By comparing this with the one-body case (2), we obtain

$$E \geqslant \mathcal{E} \geqslant Nm\sqrt{\frac{1+\sqrt{1-\gamma v^2}}{2}}, \qquad \gamma v^2 < 1. \tag{10}$$

Figure 1 illustrates the improvement by the lower bound (10) over the previously available simple lower bound (5).

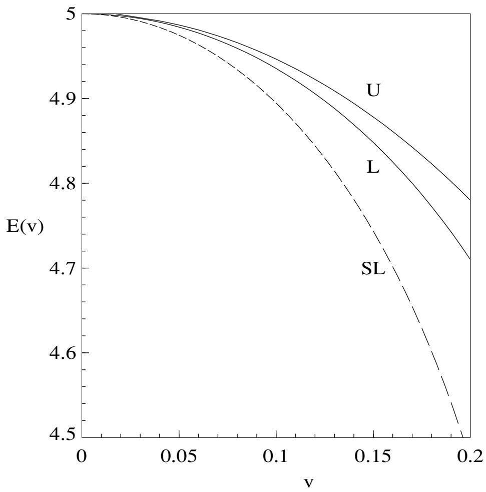

Figure 1. Energy bounds $E(v)$ (in dimensionless units) for the semirelativistic N-body problem with $N = 5$ and mass $m = 1$, as functions of the dimensionless coupling parameter v. The figure shows the simple lower bound (SL), the improved lower bound (L), and a scale-optimized Gaussian upper bound (U).

6. Gaussian upper energy bound

In order to find an upper bound, we follow [10] and use a Gaussian wavefunction, which we write in the form

$$\Phi(\boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \ldots, \boldsymbol{\rho}_N) = C \exp\left(-\frac{\alpha}{2}\sum_{i=2}^{N}\rho_i^2\right),$$

where $\alpha > 0$, while C guarantees the normalization of Φ. The boson symmetry of the trial function allows us to write $E \leqslant E_G = (\Phi, H\Phi)$, where we have

$$E_G = \left(\Phi, \left[N\sqrt{\mathbf{p}_N^2 + m^2} + \gamma V(|\mathbf{r}_1 - \mathbf{r}_2|)\right]\Phi\right) \tag{11}$$

$$= \left(\Phi, \left[N\sqrt{\lambda\boldsymbol{\pi}_N^2 + m^2} + \gamma V(|\sqrt{2}\boldsymbol{\rho}_2|)\right]\Phi\right). \tag{12}$$

The symmetry of the Gaussian function in the relative coordinates and the factoring property allow us to replace $\boldsymbol{\pi}_N$ by $\boldsymbol{\pi}_2 \equiv \mathbf{p}$, and, setting $\mathbf{r} \equiv \boldsymbol{\rho}_2$, we find explicitly

$$E \leqslant E_G = N(\phi, \sqrt{\lambda p^2 + m^2}\phi) + \gamma(\phi, V(\sqrt{2}r)\phi),$$

where $\phi(r) = (\alpha/\pi)^{3/4}\exp\left(-\frac{1}{2}\alpha r^2\right)$. The kinetic-energy expectation value may be expressed in terms of the modified Bessel function of the second kind $K_1(x)$ [26]. Evaluating the integral, for convenience, in momentum space, yields

$$(\phi, \sqrt{\lambda p^2 + m^2}\phi) = \frac{2m}{\mu}\sqrt{\frac{2}{\pi}}\, g\left(\frac{\mu^2}{4}\right),$$

where $\mu = m\{2N/[(N-1)\alpha]\}^{1/2}$, while $g(x)$ is defined by

$$g(x) = x\exp(x)K_1(x) = \int_{-\infty}^{+\infty} \mathrm{d}t\, t^2\sqrt{2x + t^2}\exp(-t^2).$$

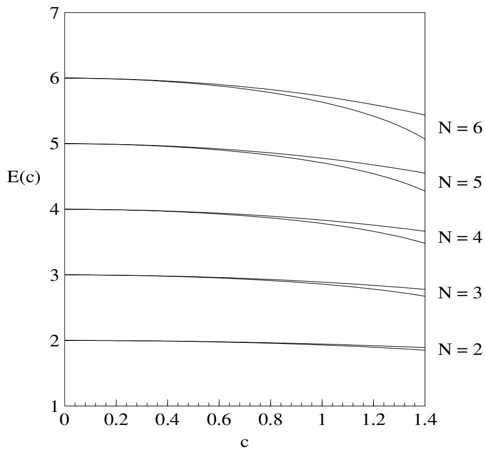

Figure 2. Upper and lower bounds to the ground-state energy $E(c)$ (in dimensionless units) of the gravitational N-particle problem, $N = 2, 3, \ldots, 6$, with mass $m = 1$, as functions of the dimensionless coupling parameter c.

The potential-energy expectation value reads for the case of a gravitational pair-interaction potential $V(r) = -v/r$:

$$(\phi, V(\sqrt{2}r)\phi) = -\frac{Nmv}{\mu}\sqrt{\frac{2}{\pi\gamma}}.$$

With the expectation values in this form, we may regard the parameter μ as a variational parameter. We arrive at

$$E \leqslant Nm\sqrt{\frac{2}{\pi}} \min_{\mu>0}\left[\frac{2g(\mu^2/4) - \sqrt{\gamma}v}{\mu}\right], \qquad N \geqslant 2. \tag{13}$$

The analytical fact that the function $[2g(\mu^2/4) - a]/\mu$ has a minimum only if $a < 2$ entails the constraint $v < 2/\sqrt{\gamma}$. Figure 1 confronts the two lower bounds (5) and (10) with this 'scale-optimized Gaussian' variational upper bound.

7. Results and conclusion

Any consideration of an arbitrarily large number, N, of self-gravitating bosons, that is, the inspection of the limit $N \to \infty$, obliges us to force the gravitational coupling v to decrease for increasing N in some appropriate manner, in order to avoid the ('relativistic') collapse $E \to -\infty$ for $N \to \infty$. Keeping v fixed, inevitably implies collapse [27]. Consequently, following Lieb and Yau [14], let v diminish, with increasing N, according to $v = c/N$, where c is some *constant* gravitational interaction parameter. In this case the term $(N-1)^2v^2$ in the simple lower bound (5) will be replaced by λ^2c^2 (with $c < 1$), the improved lower bound (10) is modified by replacing γv^2 by $\frac{1}{2}\lambda c^2$ (with $c < \sqrt{2}$) and in the Gaussian upper bound (13) we must substitute $\sqrt{\lambda/2}c$ (with $c < 2\sqrt{2}$) for $\sqrt{\gamma}v$. Figure 2 shows resulting energy curves $E(c)$ for $N = 2, 3, \ldots, 6$ and $m = 1$. For the lower bounds we require for the dimensionless parameter $c < \sqrt{2}$; the upper bounds are valid, for all N, if $c < 2\sqrt{2}$.

Some special cases are of interest. For small coupling c, both bounds are parabolic in shape. Explicitly, they read

$$Nm\left(1-\frac{\lambda c^2}{16}\right) \leqslant E(c) \leqslant Nm\left(1-\frac{\lambda c^2}{6\pi}\right). \tag{14}$$

These energy bounds coincide with previous findings for the corresponding nonrelativistic problem; cf equation (4.2) of [12]. Meanwhile, for all couplings $c < \sqrt{2}$ (such that both constraints on c are satisfied), in the limit $N \uparrow \infty$ we have $\lambda \to 1$, and the general results (10) and (13) provide bounds immediately to $E(c)/Nm$ for arbitrarily large N.

The energy spectrum of a system of nonrelativistic identical bosons experiencing any attractive pair interaction is bounded from below by the lowest energy of a specially scaled one-body problem [11]. For the harmonic oscillator, this bound yields the exact energy; such results are facilitated by use of (orthogonal) relative coordinates. Tight energy bounds have recently been obtained [10] for semirelativistic problems with oscillator pair interactions. In this analysis we have constructed both lower and upper bounds to the lowest energy of a semirelativistic system of identical gravitating bosons. Our bounds reduce to their nonrelativistic counterparts in the limit of weak coupling.

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Appendix A

For definiteness, we consider the relative momenta $\boldsymbol{\pi}_i$ and $\boldsymbol{\pi}_j$ with $i, j > 1$. Since $[\pi] = B[p]$, we have

$$\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j = \left(\sum_k B_{ik}\mathbf{p}_k\right) \cdot \left(\sum_k B_{jk}\mathbf{p}_k\right) = \sum_k B_{ik}B_{jk}\left(\mathbf{p}_k^2\right) + \sum_{k\neq l} B_{ik}B_{jl}\left(\mathbf{p}_k \cdot \mathbf{p}_l\right). \tag{A.1}$$

By using the boson symmetry of the wavefunction we have

$$\langle \boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j \rangle = \left(\sum_k B_{ik}B_{jk}\right)\left\langle \mathbf{p}_1^2\right\rangle + \left(\sum_{k\neq l} B_{ik}B_{jl}\right)\langle \mathbf{p}_1 \cdot \mathbf{p}_2\rangle. \tag{A.2}$$

But orthogonality of the rows of B to the first row tells us

$$0 = \sum_k B_{ik} = \left(\sum_k B_{ik}\right)\left(\sum_l B_{jl}\right) = \left(\sum_k B_{ik}B_{jk}\right) + \left(\sum_{k\neq l} B_{ik}B_{jl}\right). \tag{A.3}$$

If $i = j$, then the orthogonality of the matrix B tells us that

$$\sum_k B_{ik}^2 = 1 = -\sum_{k\neq l} B_{ik}B_{il}.$$

Meanwhile, if $i \neq j$, we know that the corresponding rows of B are orthogonal; hence, in this case, $\sum_k B_{ik}B_{jk} = 0$; thus, both of the coefficients in (A2) are zero. Hence we conclude for all $i, j > 1$

$$\langle \boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j \rangle = \delta_{ij}\left\langle \boldsymbol{\pi}_2^2\right\rangle = \delta_{ij}\left(\left\langle \mathbf{p}_1^2\right\rangle - \langle \mathbf{p}_1 \cdot \mathbf{p}_2\rangle\right). \tag{A.4}$$

The proof for the relative coordinates $\{\boldsymbol{\rho}_i\}$ is identical.

Appendix B

The vectors $\mathbf{P}$ and $\mathbf{p}$ (with $\boldsymbol{\pi}_1$ removed) define a plane; we let $\mathbf{k}$ be a unit vector perpendicular to this plane. Then, for example, we have

$$(m^2 + (\mathbf{p} + \delta\mathbf{P})^2)^{\frac{1}{2}} = |m\mathbf{k} + \mathbf{p} + \delta\mathbf{P}|. \tag{B.1}$$

In this notation the expectation of the kinetic-energy term in $\mathcal{H}$, defined in equation (8), reads

$$\langle |m\mathbf{k} + \mathbf{p} + \delta\mathbf{P}|\rangle = \langle |m\mathbf{k} + \mathbf{p} - \delta\mathbf{P}|\rangle. \tag{B.2}$$

Now, we may write

$$m\mathbf{k} + \mathbf{p} = \tfrac{1}{2}[(m\mathbf{k} + \mathbf{p} + \delta\mathbf{P}) + (m\mathbf{k} + \mathbf{p} - \delta\mathbf{P})]. \tag{B.3}$$

The triangle inequality then tells us

$$|m\mathbf{k} + \mathbf{p}| \leqslant \tfrac{1}{2}|m\mathbf{k} + \mathbf{p} + \delta\mathbf{P}| + \tfrac{1}{2}|m\mathbf{k} + \mathbf{p} - \delta\mathbf{P}|. \tag{B.4}$$

If we now look at mean values, we see from (B2) that

$$\langle |m\mathbf{k} + \mathbf{p}|\rangle \leqslant \langle |m\mathbf{k} + \mathbf{p} + \delta\mathbf{P}|\rangle, \tag{B.5}$$

or, equivalently

$$\langle (m^2 + p^2)^{\frac{1}{2}}\rangle \leqslant \langle (m^2 + (\mathbf{p} + \delta\mathbf{P})^2)^{\frac{1}{2}}\rangle. \tag{B.6}$$

Thus we conclude that $\langle H\rangle = \langle \mathcal{H}\rangle \geqslant \langle \mathcal{H}_{N/2}\rangle$, where the reduced one-body Hamiltonian $\mathcal{H}_{N/2}$ is given by

$$\mathcal{H}_{N/2} = N(m^2 + p^2)^{\frac{1}{2}} + \gamma V(r). \tag{B.7}$$

We note the spectral inequality $\mathcal{H} \geqslant \mathcal{H}_{N/2}$ with equality only when $N = 2$. Thus $E = \langle H\rangle = \langle \mathcal{H}\rangle \geqslant \langle \mathcal{H}_{N/2}\rangle$.

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